Uber Rocket Calculus Problem

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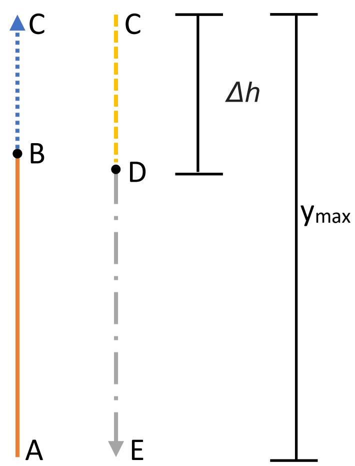
Description

One calm afternoon Calculus Cam decides to launch Hamster Huey into the air using a model rocket. The rocket is launched straight up off the ground, from rest. The rocket engine is designed to burn for specified time while producing non-constant net acceleration given by the equations below. After the engine stops the rocket continues upward in free-fall. A parachute opens after the rocket falls a specified vertical distance from its maximum height. When the parachute opens, assume the rocket instantly stops, and then increases speed to a terminal velocity given by the equation below. Assume the air resistance affects the rocket only during the parachute stage.

Diagram



Givens

$t_{AB} = 5.3s$

$\Delta h = -112m$

$y_A = 0m$

$a_{BD} = -9.80m/s^2$

$v_A = 0m/s$

$a_{AB}[t] = -0.9t^2 + 20$

$v_C = 0m/s$

$v_p[t] = -12\,(1 - e^{\frac{t}{9}})$

$t_D = 0m/s$

Strategy

In order to solve this problem, it will be divided into 3 stages: AB, BD, DE. I will find the times of each of these sections, then sum them up. As acceleration is given as a function of time for section AB, the indefinite integral can be taken of that equation to find the velocity for that same time section. The indefinite integral generates some constant c, which is equal to the velocity at A, which happens to be 0 m/s. Similarly, the constant c for the position versus time equation is also equal to 0, and both can be ignored. The rocket is in free fall for all of stage 2, so a max height can be found when velocity equals 0, which is at point C. The given drop distance can be taken and subtracted from the max height to find the height at point D. Time for stage BD can be found by using the final velocity versus initial velocity equation. For the last stage, the given velocity equation can be integrated to find a position versus time equation. The constant c can be found by setting t equal to 0 and equating the equation to the position at point D. With the constant, t can be solved for when the position equals 0, which is at point E. Finally, all three times can be summed to find the total change in time.

Stage AB:

First the equation for the velocity during AB can be found by taking the indefinite integral of the acceleration equation. Note that the velocity at A equals 0, so the constant equals 0.

$$a_{AB}[t] = -0.9t^2 + 20$$

$$v_{AB}[t] = \int (a_{AB}[t])\,dt + c$$

$$v_{AB}[t] = \int (-0.9t^2 + 20)dt + v_A$$

$$v_{AB}[t] = \frac{-0.9}{3}t^3 + 20t$$

Now substitute t_{AB} into this equation and find the velocity at B:

$$v_{AB}[5.3] = \frac{-0.9}{3}(5.3)^3 + 20(5.3)$$

$$\underline{v_B = 61.337\ m/s}$$

$$v_{AB}[t] = \frac{-0.9}{3}t^3 + 20t$$

The equation for the position versus time can be found in a similar way, which is by taking the indefinite integral of the velocity equation. Again, the constant equals 0 since the position at A is 0.

$$y_{AB}[t] = \int (v_{AB}[t])dt + c$$

$$y_{AB}[t] = \int (-0.3t^3 + 20t)dt + y_A$$

$$y_{AB}[t] = \frac{-0.3}{4}t^4 + 10t^2$$

Substitute in the time of Stage AB to find the total displacement of the stage, also known as the position at point B.

$y_{AB}[5.3] = \frac{-0.3}{4}(5.3)^4 + 10(5.3)^2$
$\underline{y_B = 221.72m}$

Stage BC

The maximum height of the system can be found at point C, where velocity equals 0. Note that in freefall the acceleration is always -9.80m/s^2.

$v_c{}^2 = v_B{}^2 + 2a_{BD}\Delta y_{BC}$
$0 = (61.337)^2 + 2(-9.80)(y_{max} - y_{b)}$
$0 = (61.337)^2 + 2(-9.80)(y_{max} - y_{b)}$
$y_{max} = \frac{(61.337)^2}{2(-9.80)} + 221.72$
$\underline{y_{max} = 413.67m}$

Stage CD

Now the position at point D can be found by using the given drop distance, Δh as well as the max height:

$\Delta h = y_B - y_{max}$
$y_D = \Delta h + y_{max}$
$y_D = -112 + 413.67$
$\underline{y_D = 301.67m}$

With the information we have we can solve for the time elapsed between points B and D:

$y_D = \frac{1}{2}a_{BD}t^2 + v_B t + y_B$

$y_D = \frac{1}{2}a_{BD}t^2 + v_B t + y_B$

$301.67 = \frac{1}{2}(-9.80)t^2 + (61.337)t + 221.72$
$-4.9t^2 + 61.337t - 79.95 = 0, SOLVER$
$\underline{t = 11.038s \text{ or } \sout{t = 1.47818s}}$

Stage DE

Given the velocity equation, a position versus time equation can be taken by taking the indefinite integral of the velocity graph. The constant can be solved for, as stated in the strategy:

$v_p[t] = -12\,(1 - e^{\frac{t}{9}})$

$y_p[t] = \int (v_p[t])\, dt + c$
$y_p[t] = \int (-12\,(1 - e^{\frac{-t}{9}}))\, dt + c$
$y_p[t] = \int (-12 + e^{\frac{-t}{9}})\, dt + c$
$y_p[t] = -12t - 108e^{\frac{-t}{9}} + c$

$y_p[0] = -12(0) - 108e^{\frac{0}{9}} + c = 301.67$
$-108 + c = 301.67$

$\underline{c = 409.67}$

With the constant solved for, we can solve for the time between points D and E, when position equals 0 at point E.

$y_p[t] = -12t - 108e^{\frac{-t}{9}} + 409.67$
$0 = -12t - 108e^{\frac{-t}{9}} + 409.67, SOLVER$

$\underline{t = 33.93s \text{ or } \sout{t = -15.3387s}}$

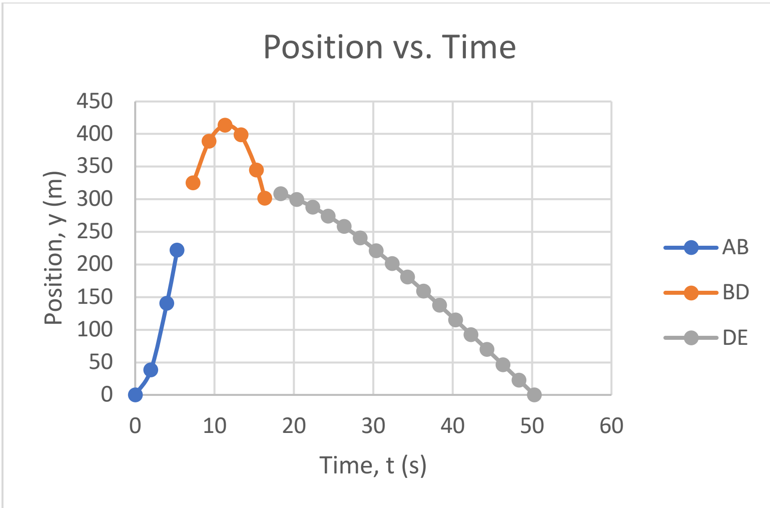
Final Sum

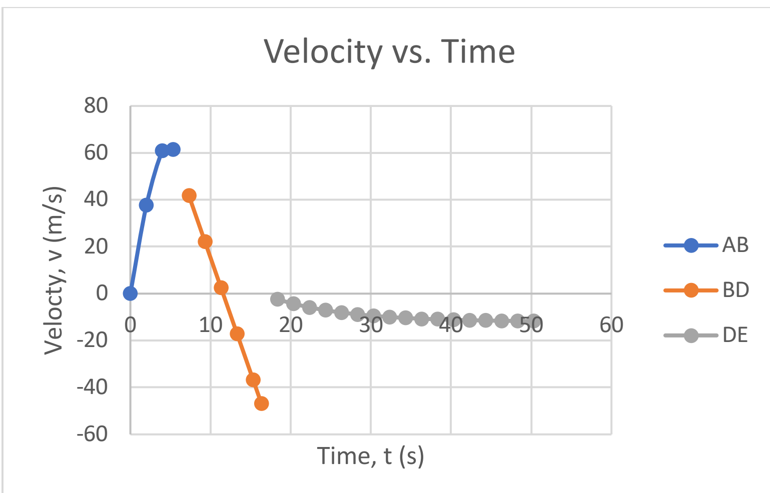
By summing all the times we have found, including the given time for Stage AB, we can find the total time elapsed.

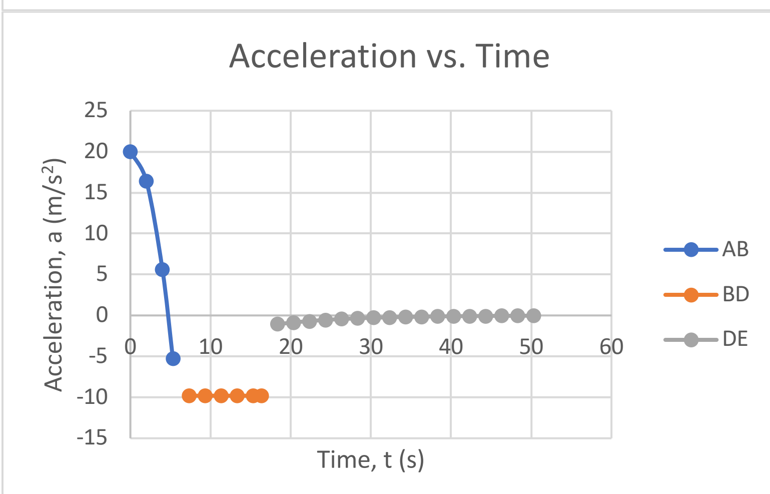
$t_{total} = t_{AB} + t_{Bc} + t_{DE}$

$$t_{total} = 5.3s + 11.038s + 33.93s$$

$$\boxed{t_{total} = 50.27s}$$


Position vs. Time
Position, y (m)
Time, t (s)
450
400
350
300
250
200
150
100
50
0
0
10
20
30
40
50
60
AB
BD
DE


Velocity vs. Time
Velocty, v (m/s)
Time, t (s)
80
60
40
20
0
-20
-40
-60
0
10
20
30
40
50
60
AB
BD
DE


Acceleration vs. Time
Acceleration, a (m/s²)
Time, t (s)
25
20
15
10
5
0
-5
-10
-15
0
10
20
30
40
50
60
AB
BD
DE